Chapter 1 Representations of Finite Groups

1.1 Linear Representatuions

Group Representations

Definition 1.1.

Let V be a vector space over the field $\mathbb{C}$ of complex numbers and let $GL(V)$ be the group of isomorphisms of V onto itself.

Suppose now G is a finite group. A linear representation of G in V is a homomorphism $\rho : G \to GL(V)$. When ρ is given, we say that V is a representation space of G, or even simply, a representation of G. ♣

In the following discussions, we restrict ourselves to the case where V has finite dimension.

Definition 1.2.

A subrepresentation of a representation V is a vector subspace W of V which is invariant under the action of G. The restriction of ρ^W of ρ to the is then an isomorphism of W onto itself. Then $\rho^W : G \to GL(W)$ a linear representation of G in W. ♣

We have a very basic but important theory of subrepresentations

Theorem 1.1.

Let $\rho : G \to GL(V)$ be a linear representation of G in V and let W be a invariant subspace of V under G. Then there exists a complement W_0 of W in V which is also invariant under G, so that $V = W \oplus W^0$. ♡

Proof For convenience, denote $\rho(g)$ as g. Let $\pi_0 : V \to W$ be a projection, we can construct the average of the map

$$\pi = \frac{1}{|G|} \sum_{g \in G} g \pi g^{-1}$$

For $w \in W$

$$g \pi_0 g^{-1} w = g(\pi_0(g^{-1} w)) = g g^{-1} w = w$$

so $\pi w = w$. Since π_0 maps V into W and g preserves W, then π is also a projection, and we have

$$\pi g w = g w = g \pi w \Rightarrow \pi g = g \pi$$

If now $x \in W^0$, $\forall g \in G$ we have

$$\pi(gx) = g(\pi x) = 0$$

which shows that W^0 is invariant under G

Faithful Representations: A linear representation in which different elements g of G are represented by distinct linear mappings $\rho(g)$. In more abstract language, this means that the group homomorphism $\rho : G \to GL(V)$ is injective.

Regular Representations: A linear representation afforded by the group action of G on itself by translation.

Unitary Representations: A linear representation ρ of G on a complex Hilbert space H such that $\forall g \in G$, $\rho(g)$ is a unitary operator, denoted R_G.

Trivial Representations: A representation (V, ρ) of a group G on which all elements of G act as the identity mapping of V.

Group Algebras and Regular Representations

Definition 1.3.

The group algebra A_G of G is the set of formal linear combinations

$$v = \sum_{g \in G} c(g)g$$

♣

All $g \in G$ are considered to be linearly independent, which can be viewed as basis of A_G Hence, we have

$$\dim A_G = |G| \tag{1.1}$$

Definition 1.4.

The regular representation $R : G \to GL(A_G)$ is the representation of the group on its own algebra, defined by

$$R(g)v = gv = \sum_{s \in G} c(s)(gs)$$

♣

Direct Sums and Tensor Products

Let (V_1, ρ_1) and (V_2, ρ_2) be two linear representations of a group G. Their direct sum is the direct sum of vector spaces $V_1 \oplus V_2$ with the linear action of G uniquely determined by the condition that

$$\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1} \oplus \rho_{V_2})(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2) \tag{1.2}$$

and the matrix representation can be written in diagnolized form

$$\rho_{V_1 \oplus V_2}(g) = \begin{bmatrix} \rho_{V_1}(g) & \\ & \rho_{V_2}(g) \end{bmatrix} \tag{1.3}$$

Their tensor product is the tensor product of vector spaces $V_1 \otimes V_2$ with the linear action of G uniquely determined by the condition that

$$\rho_{V_1 \otimes V_2}(g)(v_1 \otimes v_2) = (\rho_{V_1} \otimes \rho_{V_2})(g)(v_1 \otimes v_2) = \rho_{V_1}(g)v_1 \otimes \rho_{V_2}(g)v_2 \tag{1.4}$$

To write this down more explicitly we introduce the Kronecker product of an $n \times n$ matrix A and an $m \times m$ matrix B as

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{1.5}$$

If V and W are representations, the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations.

$$\dim(V_1 \otimes V_2) = (\dim V_1)(\dim V_2) \tag{1.6}$$

$$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2 \tag{1.7}$$

Exterior Powers and Symmetric Powers

The exterior powers $\wedge^n V$(or alternating power $\mathrm{Alt}^n V$) and symmetric powers $\mathrm{Sym}^n V$ are subrepresentations of $V^{\otimes n}$.

1.2 Irreducible Representations and Schur's Lemma

Irreducible Representations

Definition 1.5.

Let $\rho : G \to GL(V)$ be a linear representation of G. We say that it is irreducible if V has no nontrivial invariant subspace. Of course if V has nontrivial invariant subspaces, it is reducible. ♣

Corollary 1.1.

Every representation is a direct sum of irreducible representations. ♡

Proof Using Thm1.1 we can decompose V into an irreducible representations and its complement W^0. If W^0 is irreducible then it is over. If not, just apply Thm1.1 again on W^0 and repeat this process until all the subrepresentations is irreducible.

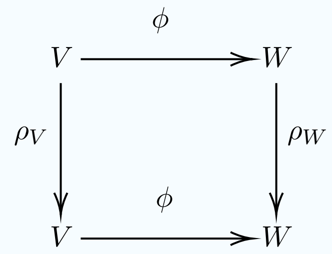
This property is called complete reducibility, which tells us that every representation can be decompose into some irreducible representations by direct sum.

Schur's Lemma

Theorem 1.2.

If V and W are irreducible representations of G and $\phi : V \to W$ is a linear map such that $\rho_V \phi = \phi \rho_W$, then

1. *Either ϕ is an isomorphism. or $\phi = 0$*
2. *If $V = W$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$, we say that ϕ is a homothety*



Proof (1) Notice that Kerϕ and Imϕ are invariant subspaces, and V and W are irreducible representations. So either Ker$\phi = V$, Im$\phi = 0$ or Ker$\phi = 0$, Im$\phi = W$, which lead to the statement that either ϕ is an isomorphism. or $\phi = 0$.

(2) If $V = W$, since $\mathbb{C}$ is algebraically closed, there exists at least one eigenvalue of ϕ, says λ. As a result, $\phi - \lambda I$ has a nonzero kernel, immediately (1) tells us that $\phi - \lambda I = 0$, so $\phi = \lambda I$.

We will call the map ϕ a **G-linear map** when we want to distinguish it from an arbitrary linear map between the vector spaces V and W.

Corollary 1.2.

All irreducible representations of an abelian group are one-dimensional.

Proof For abelian group we have

$$\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$$

Using Schur's lemma we have $\rho(g) = \lambda I$, which leads to the representation is one-dimensional.

1.3 Dual Representations

Definition 1.6.

Let G be a group and ρ is a linear representation of it on the vector space V, then the dual representation ρ^* is defined over the dual vector space V^* as follows:

$$\rho^*(g) = \rho^T(g^{-1})$$

Motivation: we want the two representations of G to respect the natural pairing $(\ ,\ )$ between V^* and V like (If I do transformations in both V and V^* using representations of the same group element, then nothing should be changed)

$$(\rho^*(g)v^*, \rho(g)w) = (v^*, (\rho^*)^T(g)\rho(g)w) = (v^*, w)$$

This in turn forces us to define the dual representation by

$$(\rho^*)^T(g)\rho(g) = I \Rightarrow \rho^*(g) = \rho^T(g^{-1})$$

Of course the representation ρ and its dual representation ρ^* have the same dimension

Now the linaer map from V to W can be written in a much more fancy way as

$$\mathrm{Hom}(V, W) = V^* \otimes W \tag{1.8}$$

which eats a vetor in V and output a vector in W. Obviously, $V^* \otimes W$ is also a representation space of G. If we view an element of $\mathrm{Hom}(V, W)$ as a linear map ϕ from V to W, we have (denote $\rho(g) = g$ for convenience)

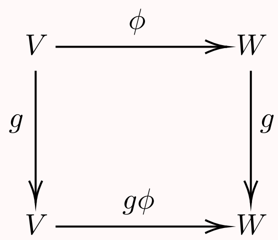
$$g\phi = g(v^* \otimes w) = (gv^*) \otimes (gw)$$

$$(g\phi)(v_0) = [(gv^*) \otimes (gw)]v_0 = [(v^*) \otimes (gw)](g^{-1}v_0) = g\phi(g^{-1}v_0) = (g\phi g^{-1})(v_0)$$

We have

$$\rho_{V^*\otimes W}(g)\phi = \rho_W(g)\phi\rho_V(g^{-1})$$

The map ϕ here may not be a G-linear map. The commutative diagram below tells us that we must have $g\phi = \phi$ if ϕ is a G-linear map.



1.4 Characters

Definition 1.7.

Let $\rho : G \to GL(V)$ be a linear representation of a finite group G in the vector space V. For each $g \in G$, define the complex valued function

$$\chi_\rho(g) = Tr(\rho_g)$$

as the character of the representation ρ

If χ is the character of a representation ρ of degree n, its essential of trace gives the properties below

1. $\chi_\rho(e) = \mathrm{dim}V = n$
2. $\chi_\rho(g^{-1}) = \bar{\chi}_\rho(g)$
3. $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$

Remarks. If ρ is irreducible, the character is called **simple**; otherwise, it is called **compound**. A function f on G satisfying identity (3), or what amounts to the same thing, $f(uv) = f(vu)$, is called a **class function**.

Proposition 1.1.

Let V and W be representations of G, then

$$\chi_{V\oplus W} = \chi_V + \chi_W \qquad \chi_{V\otimes W} = \chi_V \cdot \chi_W \qquad \chi_{V^*} = \bar{\chi}_V$$

Proof

1.5 Orthogonality Relations for Characters

Fixed Points

Let ρ be a representation of group G in the vector space V, we set

$$V^G = \{v \in V | \forall g \in G, \rho_g v = v\} \tag{1.9}$$

immediately we know that V^G is a subspace of V, we define the average as

$$\varphi = \frac{1}{|G|} \sum_{g \in G} \rho_g \tag{1.10}$$

Theorem 1.3.

The map φ is a projection of V onto V^G

♡

Proof Suppose $v = \varphi(w)$, then, $\forall h \in G$ we have

$$\rho_h v = \frac{1}{|G|} \sum_{g \in G} \rho_h \rho_g w = \frac{1}{|G|} \sum_{g \in G} \rho_{hg} w = \frac{1}{|G|} \sum_{g \in G} \rho_g w = v$$

So we found that $\text{Im}\varphi \in V^G$. Conversely, if $v \subset V^G$, then

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} \rho_g v = \frac{1}{|G|} \sum_{g \in G} v = v$$

then $V^G \subset \text{Im}(\varphi)$ and $\varphi^2 = \varphi$, so $\text{Im}\varphi = V^G$ and φ is a projection of V onto V^G.

If we just want to know the number m of copies of the trivial representation appearing in the decomposition of V

$$m = \text{dim } V^G = \text{Tr } (\varphi) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \tag{1.11}$$

If ϕ and ψ are two complex-valued functions on G, put

$$\langle \phi | \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi^*(g) \psi(g) \tag{1.12}$$

we have

$$m = \langle \chi_I | \chi_\rho \rangle \tag{1.13}$$

where I is the irreducible trivial representation where we must have $\chi_I = 1$.

Proposition 1.2.

Let ρ be a representation of group G in the vector space V, we have

$$\text{dim } V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \langle \chi_I | \chi_\rho \rangle$$

♠

Orthogonality Relations

If V and W are representations of G, then with the representation $\mathrm{Hom}(V, W)$, we have

$$\mathrm{Hom}_G(V, W) = \{\text{G-linaer maps from } V \text{ to } W\}$$

The discussion in section1.3 tells us that we must have $g\phi = \phi$ if ϕ is a G-linear map, so we have

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G \tag{1.14}$$

Using proposition1.5, we have

$$\dim \mathrm{Hom}(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \bar{\chi}_V(g)\chi_W(g) = \langle \chi_V | \chi_W \rangle \tag{1.15}$$

If V is irreducible, then, by Schur's lemma, we have

$$\dim \mathrm{Hom}_G(V, W) = \text{the multiplicity of } V \text{ in } W$$

Similarly if W is irreducible, we have

$$\dim \mathrm{Hom}_G(V, W) = \text{the multiplicity of } W \text{ in } V$$

If both V and W are irreducible, we have

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$$

Theorem 1.4.

In terms of this inner product, the characters of the irreducible representations of G are orthonormal. If χ and χ' are the characters of two nonisomorphic irreducible representations, we have

1. $||\chi||^2 = \langle \chi | \chi \rangle = 1$
2. $\langle \chi | \chi' \rangle = 0$

♡

If $V = V_1^{\oplus \alpha_1} + \cdots + V_k^{\oplus \alpha_k}$, with the V_i distinct irreducible representations, we have the following corollaries.

Corollary 1.3.

Any representation is determined by its character

♡

Proof Indeed if $V = V_1^{\oplus \alpha_1} + \cdots + V_k^{\oplus \alpha_k}$, we have

$$\chi = \sum_i \alpha_i \chi_i$$

where the χ_i are orthonormal.

This tells us that

$$\langle \chi_i | \chi \rangle = \alpha_i$$

which leads to the two following corollaries

Corollary 1.4.

If $V = V_1^{\oplus\alpha_1} + \cdots + V_k^{\oplus\alpha_k}$, *with the* V_i *distinct irreducible representations, we have*

1. *The multiplicity* α_i *of* V_i *in* V *is* $\alpha_i = \langle\chi_i|\chi\rangle$
2. *A representation* V *is irreducible if and only if* $\langle\chi|\chi\rangle = 1$

♡

1.6 Decomposition of Representations

Decomposition of Regular Representations

Proposition 1.3.

The character r_G *of a regular representation is given by*

$$r_G(e) = |G| \qquad r_G(g) = 0 \ (g \neq e)$$

♠

Proof If we use matrix representation, the regular representation will be in forms of permutation matrix. For any basis e_g and $t \neq e$, e_g, $\rho_t e_g = e_{tg} \neq e_g$, it is easy to get

$$r_G(g) = 0$$

For $t = e$, e_g, $\rho_e e_g = e_g$, we have

$$r_G(e) = \text{Tr } I = \dim R = |G|$$

Theorem 1.5.

If $V = V_1^{\oplus\alpha_1} + \cdots + V_k^{\oplus\alpha_k}$, *with the* V_i *distinct irreducible representations, we have*

1. *Every irreducible representation* V_i *is contained in the regular representation* R *with multiplicity equal to its degree* n_i
2. *The degrees* n_i *satisfy the relation*
$$\sum_i n_i^2 = |G|$$
3. *If* $g \in G$ *and* $g \neq e$, *then*
$$\sum_i n_i \chi_i(g) = 0$$

♡

Proof (1) The multiplicity is equal to $\langle r_G|\chi_i\rangle$, we have

$$\langle\chi_i|r_G\rangle = \frac{1}{|G|}\sum_{g\in G}\bar{r_G}(g)\chi_i(g) = \frac{1}{|G|}|G|\chi_i(e) = n_i$$

(2,3) By (1) we have,

$$r_G(s) = \sum_{g\in G}\langle\chi_i|r_G\rangle\,\chi_i(s) = \sum_{g\in G} n_i\chi_i(s)$$

If $s \neq e$, we have

$$r_G(s) = \sum_{g\in G} n_i\chi_i(s) = 0$$

If $s = e$, we have

$$r_G(s) = \sum_{g \in G} n_i \chi_i(s) = |G|$$

Compleness of Characters

Theorem 1.6.
Let $f : G \to \mathbb{C}$ be any function on the group G, and for any representation V of G set

$$\phi_{f,V} = \sum_{g \in G} f(g)\rho_V(g)$$

a function from $V \to V$. Then $\phi_{f,V}$ is G-linear for all V if and only if f is a class function. ♡

Proof For $\forall h \in G$ and $\forall v \in V$, we have

$$\phi_{f,V}\rho_V(h)v = \sum_{g \in G} f(g)\rho_V(g)\rho_V(h)v = \sum_{g \in G} f(g)\rho_V(gh)v$$

substituting hgh^{-1} for g we have

$$\phi_{f,V}\rho_V(h)v = \sum_{g \in G} f(hgh^{-1})\rho_V(hg)v = \rho_V(h)\sum_{g \in G} f(hgh^{-1})\rho_V(g)v$$

If f is class function we have $f(hgh^{-1}) = f(g)$ which leads to

$$\phi_{f,V}\rho_V(h)v = \rho_V(h)\sum_{g \in G} f(g)\rho_V(g)v = \rho_V(h)\phi_{f,V}v$$

so we have

$$\phi_{f,V}\rho_V(h) = \rho_V(h)\phi_{f,V}$$

Conversely, if $\phi_{f,V}$ is G-linear, we have

$$\sum_{g \in G} f(g)\rho_V(gh) = \sum_{g \in G} f(g)\rho_V(hg)$$

substituting hgh^{-1} for g on the left-hand side we have

$$\sum_{g \in G} f(hgh^{-1})\rho_V(hg) = \sum_{g \in G} f(g)\rho_V(hg)$$

The equation holds for $\forall h \in G$ so we must have

$$f(hgh^{-1}) = f(g)$$

Corollary 1.5.
The characters of irreducible representations of G $\{\chi_i\}_{i=1}^k$ form an orthonormal basis for $Cl(G)$ (Vector space of all the calss function) ♡

Proof Suppose $f : G \to \mathbb{C}$ is a class function and $\langle f|\chi\rangle = 0$ for an irreducible representations V, we must show that $f = 0$. So consider the G-linaer map

$$\phi_{f,V} = \sum_{g \in G} f^*(g)\rho_V(g)$$

By Schur's lemma, we have $\phi_{f,V} = \lambda I$. If $n = \dim V$, then

$$\lambda = \frac{\text{Tr}\,\phi_{f,V}}{n} = \frac{|G|}{n}\sum_{g\in G} f^*(g)\chi(g) = \frac{|G|}{n}\langle f|\chi\rangle = 0$$

Then $\phi_{f,V} = 0$ for any irreducible representation V of G. In particular, this will be true for the regular representation R. But in R the elements $g \in G$, thought of as elements of End(R), are linearly independent.

$$\phi_{f,R}e_h = \sum_{g\in G} f^*(g)\rho_R(g)e_h = \sum_{g\in G} f^*(g)e_g = 0$$

so $f^* = f = 0$ and the proof is complete.

Corollary 1.6.

The number of irreducible representations of G is equal to the number of conjugacy classes of G

♡

Proof The class function yeilds

$$f(hgh^{-1}) = f(g)$$

So all elements in the same conjugacy calss has the same character.

Corollary 1.7.

Let $g \in G$, and let $c(g)$ be the number of elements in the conjugacy class of g

1. *we have*

$$\sum_i = \chi_i^*(g)\chi_i(g) = \frac{|G|}{c(g)}$$

2. *For $s \in G$ not conjugate to g, we have*

$$\sum_i = \chi_i(g)^*\chi_i(h) = 0$$

♡

Proof Let f_s be the function equal to 1 on the class of s and equal to 0 elsewhere.

$$f_s(g) = \begin{cases} 1 & g \text{ in the class of } s \\ 0 & \text{Otherwise} \end{cases}$$

Since it is a class function, it can be written as

$$f_s = \sum_i \langle\chi_i|f_s\rangle\,\chi_i \quad \text{with } \langle\chi_i|f_s\rangle = \frac{c(s)}{|G|}\chi_i^*(s)$$

then, for g in the class of s we have $f_s(g) = 1$ and

$$\sum_i = \chi_i^*(s)\chi_i(g) = \sum_i \chi_i^*(s)\chi_i(s) = \frac{|G|}{c(s)}$$

for g not in the class of s we have $f_s(g) = 0$ and

$$\sum_i = \chi_i^*(s)\chi_i(g) = 0$$

1.7 Induced Representations

Induced Representations

Let G be a group and ρ is a linear representation of it on the vector space V. Let H be a subgroup of G, then the representation can be naturally restricted on H to get a representation of H, denoted as ρ_H. Conversely if we have a representation of H, how can we get the representation of G?

Let $s \in G$, the vector space

$$\rho_s W = \{\rho_s w | w \in W\} \tag{1.16}$$

depends only on the left coset sH. Indeed, if we replace s by st, with $t \in H$, we have

$$\rho_{st} W = \rho_s \rho_t W = \rho_s W \tag{1.17}$$

If σ is a left coset of H, we can thus define a subspace W_σ of V to be $\rho_s W$ for any $s \in \sigma$.

Definition 1.8.
We say that the representation ρ of G in V is induced by the representation θ of H in W if

$$V = \bigoplus_{\sigma \in G/H} W_\sigma$$

♣

In this case we write

$$V = \text{Ind}_H^G W = \text{Ind } W \qquad W = \text{Res}_H^G V = \text{Res } V \tag{1.18}$$

Notice that

$$\dim W_\sigma = \dim(\rho_s W) = \dim W \tag{1.19}$$

We have

$$\dim V = [G : H] \cdot \dim W \tag{1.20}$$

Existence and Uniqueness of Induced Representations

We claim that, given a representation W of H, such V exists and is unique up to isomorphism

Theorem 1.7.
Let W be a representation of H, U a representation of G, and suppose $V = Ind\, W$. Then any H-module homomorphism $\phi : W \to U$ extends uniquely to a G-module homomorphism $\phi' : V \to U$

$$Hom_H(W, Res\, U) = Hom_G(Ind\, W, U)$$

In particular, this universal property determines $Ind\, W$ up to canonical isomorphism ♡

Proof define ϕ' on W_σ as

$$W_\sigma \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\phi} U \xrightarrow{g_\sigma} U, \qquad \phi' : W_\sigma \to U$$

which is independent of the representative g_σ for σ since ϕ is H-linear

Characters of Induced Representations

Note that $g \in G$ maps W_σ to $W_{g\sigma}$, so the trace is calculated from those cosets

$$\chi_V(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) \quad (\forall s \in \sigma)$$