Chapter 1 Group Structures

1.1 Group Actions

Definition 1.1.

An action of a group G on a set S is a function $\rho : G \times S \to S$ (usually denoted by $\rho(g, x) = gx$ for convenience) such that lor all $x \in S$ and $g_1, g_2 \in G$

$$ex = x \qquad (g_1 g_2)x = g_1(g_2 x)$$

When such an action is given, we say that G acts on the set S

A very important application of group action is a group acts on itself, we have two kinds of group action

1. Left multiplication: $\rho(g, x) = gx$
2. Adjoint multiplication(Conjugation): $\rho(g, x) = gxg^{-1}$

Theorem 1.1.

Let G be a group that acts on a set S, we have

1. *Equivalence class: $x \sim x' \Leftrightarrow gx = x'$ for some $g \in G$*
2. *For each $x \in S$, $G_x = \{g \in G | gx = x\}$ is a subgroup of G*

The proof is easy. In Thm1.1, the equivalence classes is called the orbits of G on S, denote as $\mathcal{O}_x$. The subgroup G_x is called the stabilizer of x.

Definition 1.2.

The definition of orbits, stabilizers and fixed points

1. *Orbit: $\mathcal{O}_x = \{gx | g \in G\}$*
2. *Stabilizer: $G_x = \{g \in G | gx = x\}$*
3. *Fixed points: $Z = \{x \in S | \forall g \in G, gx = x\}$*

By definition, the orbit of a fixed point $z \in Z$ is the fixed point itself.

Theorem 1.2. I

a group G acts on a set S

1. *For $x \in S$, the cardinal number of the orbit $\mathcal{O}_x$ is the index $[G : G_x]$*
2. *If $\mathcal{O}_{x_1}, \cdots, \mathcal{O}_{x_n} (x_i \in G)$ are the distinct orbits of S, then*

$$|S| = \sum_i |\mathcal{O}_{x_i}| = \sum_i [G : G_{x_i}]$$

Proof Let $g, h \in G$. Since

$$gx = hx \Leftrightarrow h^{-1}gx = x \Leftrightarrow h^{-1}g \in G_x \Leftrightarrow gG_x = hG_x$$

it follows that the map given by $gG_x \to gx$ is a well-defined bijection of the set of cosets of G_x onto the orbit $\mathcal{O}_x$. Hence $|\mathcal{O}_x| = [G : G_x]$. Furthermore, by applying Lagrange Thm

$$|\mathcal{O}_x| = [G : G_x] = |G|/|G_x| \Rightarrow |G| = |\mathcal{O}_x||G_x|$$

Theorem 1.3.

If a group G acts on a set S, then this action induces a homomorphism $\phi : G \to A(S)$, where $A(S)$ is the group of all permutations of S.

♡

Theorem 1.4.

Let G be a p-group that acts on set S,then

$$|S| \equiv |Z(G)| \mod p$$

♡

Proof Because G is a p-group, so all the subgroups of G is p-group, then all the stabilizers G_x are p-groups. Using Thm1.1, if $\mathcal{O}_{x_1}, \cdots, \mathcal{O}_{x_n} (x_i \in G)$ are the distinct orbits of S and $|\mathcal{O}_{x_i}| > 1$, we have

$$|S| = |Z(G)| + \sum_i [G : G_{x_i}]$$

because

$$[G : G_{x_i}] \equiv 0 \mod p$$

then

$$|S| \equiv |Z(G)| \mod p$$

1.2 Conjugacy Class

Now we begin to consider the group acts on itself, which means $S = G$. The left multiplication is trivial, the only one orbit is group G itself, and the stabilizer is $\{e\}$.

Theorem 1.5.

If G is a group, then there is a monomorphism $\phi : G \to A(S)$. Hence every group is isomorphic to a group of permutations. In particular every finite group is isomorphic to a subgroup of S_n with $n = |G|$

♡

Proof Let G act on itself by left translation and apply Thm1.1 to obtain a homomorphism.

The much more special case is adjoint multiplication, $\rho(g, x) = gxg^{-1}$. The equivalence class now becomes conjugacy calss, the stabilizer becomes centralizer and the fixed points becomes center.

Definition 1.3.

Let G be a group and H a subgroup of G. If H acts on G by conjugation, for $x \in G$, we define

1. *Conjugacy calss of x in H:* $[x]_H = \{hxh^{-1}|h \in H\}$
2. *Centralizer of x in H:* $C_H(x) = \{h \in H|hxh^{-1} = x\}$
3. *Center of H:* $C(G) = \{x \in G|\forall g \in G, gxg^{-1} = x\}$

♣

We immediately get the conjugate version of Thm1.1

Corollary 1.1.

Let G be a group

1. *For each $g \in G$, conjugation by g induces an automorphism of G.*
2. *There is a homomorphism $\phi : G \to Aut(G)$ whose kernel is $C(G)$*

♡

Proof (1) If G acts on itself by conjugation, then for each $g \in G$, the map $\varphi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$ is a bijection by the proof of Thm1.1. It is easy to see that φ_g is also a homomorphism and hence an automorphism.

(2) By (1) we have $\varphi_g \in \text{Aut}(G)$, clearly

$$g \in \text{Ker}\phi \Leftrightarrow \varphi_g = \text{id} \Leftrightarrow \forall x \in G, gxg^{-1} = x \Leftrightarrow g \in C(G)$$

Corollary 1.2.

Let G be a finite group and K a subgroup of G

1. *The number of elements in the conjugacy class of $x \in G$ is $[G : C_G(x)]$*
2. *if$[x_1], \cdots, [x_n] (x_i \in G)$ are the distinct conjugacy classes of G, then*

$$|G| = \sum_i |[x_i]| = \sum_i [G : C_G(x_i)]$$

♡

1.3 Group Actions on Subsets of Groups

Now we move Further for group G acts on set S of some subsets of G.

Theorem 1.6.

Let H be a subgroup of group G and let G act on the set S of all left cosets of H in G by left translation. Then the kernel of the induced homomorphism $\phi : G \to A(S)$ is contained in H.

♡

Proof The induced homomorphism $\phi : G \to A(S)$ is given by $\phi(g) = \tau_g$, where $\tau_g(xH) = gxH$. If $g \in \text{Ker}\phi$, then $\tau_g = \text{id}_S$ and for $\forall x \in G$ we have

$$\tau_g(xH) = gxH = xH$$

In particular for $x = e$ we have

$$gH = H \Rightarrow g \in H$$

Definition 1.4.

Let G be a group and H a subgroup of G. If H acts on S by conjugation, for $K \in S$, we define the normalizer of K in H as $N_H(K) = \{h \in H | hKh^{-1} = K\} = \{h \in H | hK = Kh\}$

♣

1.4 Automorphisms and Semidirect Product

Automorphisms

An automorphism in the form of conjugate is called an inner automorphism, and the remaining automorphisms are said to be outer

Semidirect Product

Let N be a normal subgroup of G. Each element $g \in G$ defines an automorphism of N, $n \to gng^{-1}$, and this defines a homomorphism

$$\theta : G \to \text{Aut}(N), \quad g \mapsto i_g|_N \tag{1.1}$$

If there exists a subgroup Q of G such that $G \to G/N$ maps Q isomorphically onto G/N, then we can reconstruct G from N, Q, and the restriction of θ to Q. Indeed, an element g of G can be written uniquely in the form

$$g = nq \tag{1.2}$$

If $g = nq$ and $g' = n'q'$, then

$$gg' = nqn'q' = n(qn'q^{-1})qq' = n\theta_q(n') \cdot qq' \tag{1.3}$$

Further more, θ is the identity map iff that Q, N are normal subgroups

Proof Suppose that Q, N are normal subgroups, then

$$qnq^{-1} = n' \Rightarrow q(nq^{-1}n^{-1}) = n'n^{-1} \Rightarrow qq' = n'n^{-1} = e \Rightarrow \theta_q(n) = qnq^{-1} = n$$

Suppose that θ is the identity map, then

$$qnq^{-1} = n \Rightarrow qn = nq \Rightarrow nqn^{-1} = q$$

So, under the condition, the semidirect product becomes direct product.

Notes: If we have a subgroup $Q \cong G/N$, then we can use semidirect product to construct group G

Example 1.1 (D_n, A_n)

(a) In D_n, all the ratation forms $\langle R \rangle$ and all the reflection forms $r\langle R \rangle$ with the relation

$$R^n = r^2 = (Rr)^2 = e \quad \Rightarrow \quad rRr = R^{-1}$$

So we have

$$D_n = \langle R \rangle \rtimes \langle r \rangle = C_n \rtimes C_2 \qquad \theta_r(R^n) = R^{-n}$$

(b) Notice that A_n is a normal subgroup of S_n

$$C_2 = \{e, (12)\} \cong S_n/A_n \Rightarrow S_n = A_n \rtimes C_2$$