Chapter 1 Lie Algebra Structure

1.1 Lie Algebra Structure

Definition 1.1.

For a Lie algebra $\mathfrak{g}$ we have the following definitions

1. *A subset $\mathfrak{a} \subset \mathfrak{g}$ is called a (Lie) sub-algebra if $\mathfrak{a}$ is a linear subspace and closed under Lie braket.*
2. *A sub-algebra $\mathfrak{a}$ is called Abelian if $[X, Y] = 0$ for $\forall X, Y \in \mathfrak{a}$*
3. *A sub-algebra $\mathfrak{a}$ is an ideal if $[X, Y] \in \mathfrak{a}$ for $\forall X \in \mathfrak{g}$ and $\forall Y \in \mathfrak{a}$*
4. *The derived series $\{D^k\mathfrak{g}\}$ of $\mathfrak{g}$ is defined by $D^1 = [\mathfrak{g}, \mathfrak{g}]$ and $D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$*
5. *$\mathfrak{g}$ is called solvable if $D^k\mathfrak{g} = 0$ for some k*

♣

Definition 1.2.

A Lie algebra $\mathfrak{g}$ is called

1. *simple if it contains no non-trivial ideals*
2. *semi-simple if it contains no non-zero solvable ideals*

♣

Theorem 1.1.

(1) A Lie algebra $\mathfrak{g}$ is semi-simple $\Leftrightarrow$ $\mathfrak{g}$ has no non-zero Abelian ideals

(2) If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are solvable ideals, then so is $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$

♡

Proof (1) Because $\mathfrak{g}$ is semi-simple, then all the non-zero ideal $\mathfrak{h}$ of $\mathfrak{g}$ is unsovlable. We have $D^1\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \neq 0$, which means that $\mathfrak{g}$ has no non-zero Abelian ideals. Conversly, it is straightforward to show (from the Jacobi identity) that for an ideal $\mathfrak{h}$ of $\mathfrak{g}$ the entire derived series $\{D^k\mathfrak{h}\}$ consists of ideals.

(2) $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ clearly is an ideal and all we need to show is that it is solvable.

Definition 1.3.

The sum of all solvable ideals in $\mathfrak{g}$ is called the radical rad($\mathfrak{g}$) of $\mathfrak{g}$

♣

Theorem 1.2.

A Lie algebra $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus_s \mathfrak{a}$$

where $\mathfrak{a}$ is semi-simple

♡

Theorem 1.3.

A semi-simple Lie algebra $\mathfrak{g}$ *is a direct sum of simple Lie algebras* $\mathfrak{g}_i$

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$$

The direct sum refers to a direct vector space sum and the commutators $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ *for* $i \neq j$.

Combining the previous two theorems we learn that every Lie algebra $\mathfrak{g}$ can be written in the form

$$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus_s \left(\bigoplus_i \mathfrak{g}_i \right) \tag{1.1}$$

where the $\mathfrak{g}_i$ are simple and commute with each other

Consider the direct product group $G = G_1 \times G_2$ of two Lie groups G_1 and G_2. This is again a Lie group with Lie algebra $L(G) \cong T_1 G = T_1 G_1 \oplus T_1 G_2 \cong L(G_1) \oplus L(G_2)$ (The sum is direct since vectors fields on G_1 commute with those on G_2). Hence, a direct product of Lie groups leads to a direct sum of the associated Lie algebras.

1.2 The Killing Form

The Killing Form

A Lie algebra carries a symmetric bi-linear form, the Killing form, which plays an important role in analysing the structure of Lie algebras.

Definition 1.4.

The symmetric bilinear form $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ *defined by*

$$\Gamma(X, Y) = \mathrm{Tr}[\mathrm{ad}_X \mathrm{ad}_Y]$$

is called the Killing form of $\mathfrak{g}$

We can use the representation matrices of the adjoint representation to compute the Killing form $\gamma_{ij} = \Gamma(X_i, X_j)$ relative to a basis $\{X_i\}$

$$\gamma_{ij} = \Gamma(X_i, X_j) = \mathrm{Tr}(\mathrm{ad}_{X^i} \mathrm{ad}_{X^j}) = (\mathrm{ad}_{X_i})_l{}^k (\mathrm{ad}_{X_j})_k{}^l = c_{il}{}^k c_{jk}{}^l \tag{1.2}$$

Then, the Killing form of two Lie algebra elements $T = v^i X_i$ and $S = w^j X_j$ can be written as $\Gamma(T, S) = \gamma_{ij} v^i w^j$

Proposition 1.1.

The Killing form satisfies

$$\Gamma(X, \mathrm{ad}_Z Y) = -\Gamma(\mathrm{ad}_Z X, Y)$$

Proof

$$\begin{aligned}\Gamma(X, \mathrm{ad}_Z Y) &= \Gamma(X, [Z, Y]) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_{[Z,Y]}) = \mathrm{Tr}(\mathrm{ad}_X [\mathrm{ad}_Z, \mathrm{ad}_Y]) \\ &= \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Z \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y \mathrm{ad}_Z) \\ &= \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Z \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_Z \mathrm{ad}_X \mathrm{ad}_Y) \\ &= -\mathrm{Tr}([\mathrm{ad}_Z, \mathrm{ad}_X] \mathrm{ad}_Y) = -\mathrm{Tr}(\mathrm{ad}_{[Z,X]} \mathrm{ad}_Y) = -\Gamma([Z, X], Y) \\ &= -\Gamma(\mathrm{ad}_Z X, Y)\end{aligned}$$

Kiling Form and Lie Algebra Structure

The Killing form Γ is called (non-degenerate) if $\Gamma(X, Y) = 0$ for all $X \in \mathfrak{g}$ implies that $Y = 0$. Equivalently, Γ is non-degenerate if the matrix (Γ_{ij}) is invertible. The Killing form can be used to decide whether a Lie algebra is semi-simple.

Theorem 1.4.

(1) A Lie algebra $\mathfrak{g}$ is semi-simple iff Γ is non-degenerate.

(2) If G is compact, then the Killing form Γ on $L(G)$ is negative semi-definite ♡

Proof Fxxk!

Killing Form and Structure Factors

Practical applications are often formulated in terms of a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$. Structure factors ${c_{ij}}^k$ and the Killing form $\gamma_{ij} = {c_{ik}}^l {c_{jl}}^k$. If $\mathfrak{g}$ is semi-simple then γ_{ij} is invertible and we can also introduce its inverse γ^{ij}. It is useful to translate some of the previous results into this language.

The Jacobi identity $[[X_i, X_j], X_k] + [[X_j, X_k], T_i] + [[X_k, X_i], X_j] = 0$ together with the commutation relations $[X_i, X_j] = {c_{ij}}^k X_k$ translates into the relation

$$ {c_{ij}}^l {c_{kl}}^n + {c_{jk}}^l {c_{il}}^n + {c_{ki}}^l {c_{jl}}^n = 0 \tag{1.3}$$

Of course the structure factors are anti-symmetric in the first two indices, so

$$ {c_{ij}}^k + {c_{ji}}^k = 0 \tag{1.4}$$

To get to a stronger statement, we use the Killing metric to lower and raise the last index

$$ c_{ijk} = \gamma_{lk} {c_{ij}}^l \tag{1.5}$$

and with a short calculation we can show that c_{ijk} is totally anti-symmetric.

Quadratic Casimir

If $\mathfrak{g}$ is semi-simple we can define the quadratic Casimir operator as

$$ C = \gamma^{ij} X_i X_j \tag{1.6}$$

Its relevance is that it commutes with the entire Lie algebra

Proposition 1.2.

The Casimir operator satisfies $[C, X] = 0$ *for all* $X \in \mathfrak{g}$

Proof

$$[C, X_k] = \gamma^{ij}[X_iX_j, X_k] = \gamma^{ij}X_i[X_j, X_k] + \gamma^{ij}[X_i, X_k]X_j$$
$$= \gamma^{ij}c_{jk}{}^{l}X_iX_l + \gamma^{ij}c_{ik}{}^{l}X_lX_j = \gamma^{ij}c_{ik}{}^{l}(X_iX_l + X_lX_i)$$
$$= \gamma^{ij}\gamma^{mn}c_{ikn}(X_iX_m + X_mX_i) = 0$$

Schur's Lemma can be applied at the level of the algebra, so if the Lie algebra is irreducible then $C = \lambda 1$.

Physics Conventions

Assuming that the Lie group G is compact and that its algebra $L(G)$ is simple. Form Thm1.2 we learn that this Killing form is non-degenerate and negative semi-definite and, hence, non-degenerate and negative definite(non-degenerate means invertible, which brings negative semi-definite to negative definite). This means we can choose a basis $\{X_i\}$ of $L(G)$ such that

$$\gamma_{ij} = -\delta_{ij} \Rightarrow c_{ijk} = \gamma_{kl}c_{ij}{}^{l} = -c_{ij}{}^{k} \tag{1.7}$$

As a result the structure constants $c_{ij}{}^{k}$ which appear in the commutation relations are completely anti-symmetric, in the same way as their lower index counterparts c_{ijk}

Now consider an irreducible representation $r : L(G) \to \text{End}(V)$, where $d = \dim(r)$ and write the representation matrices of the basis as $T_i^{(r)} = r(T_i)$. For the Casimir $C^{(r)}$ in the representation r we have

$$C^{(r)} = -\delta_{ij}T_i^{(r)}T_j^{(r)} = -\sum_i (T_i^{(r)})^2 \tag{1.8}$$

Using Schur's Lemma we have

$$C^{(r)} = -\sum_i (T_i^{(r)})^2 = C(r)I \tag{1.9}$$

1.3 The Cartan Subalgebras

Cartan Subalgebras

Definition 1.5.

Let $\mathfrak{g}$ *be a semi-simple Lie algebra, a Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *satisfies*

1. *For all* $H \in \mathfrak{h}$*, the adjoint actions* ad_H *can be diagonalised simultaneously, and we can say that* $\mathfrak{h}$ *is diagonalisable*
2. $\mathfrak{h}$ *is maximal, if, for some* $X \in \mathfrak{g}$*, we have* $[X, H] = 0$ *for all* $H \in \mathfrak{h}$*, then* $X \in \mathfrak{h}$

Of course there are a number of things to clarify about this definition. We need to worry about the existence and construction of the Cartan subalgebra, and, since it turns out it is not unique, about whether dim($\mathfrak{h}$) is well-defined. But never mind, mathematicians have shown that $\mathfrak{h}$ always exists and that the rank, rank($\mathfrak{g}$) = dim($\mathfrak{h}$), is indeed well-defined.

The first thing we need to know is that $\mathfrak{h}$ is Abelian. Since $\mathfrak{h}$ is diagonalisable we have

$$0 = [\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}] = \mathrm{ad}_{[H_1,H_2]} \tag{1.10}$$

which force $[H_1, H_2] = 0$ because $\mathfrak{g}$ is semi-simple.

Proposition 1.3.

Let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be a complex semisimple Lie algebra of a complex matrix Lie group. And let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{k}$. Define $\mathfrak{h} \subset \mathfrak{g}$ by

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$

If $\mathfrak{g}$ is a complex semisimple Lie algebra, the rank of $\mathfrak{g}$ is the dimension of any Cartan subalgebra. For the rest of this chapter, we assume that we have chosen a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ and a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and we consider the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$.

1.4 Roots

$\mathfrak{h}$ is diagonalisable means that we can study the simultaneous eigenvectors $X \in \mathfrak{g}$ which satisfy the equation

$$\mathrm{ad}_H X = \alpha_H X \quad \text{for all } H \in \mathfrak{h} \tag{1.11}$$

We can define α as a linear functional such that eats an element in $\mathfrak{h}$ and output an eigenvalue. $\alpha_H = \langle \alpha | H \rangle$ depends linearly on H.

Definition 1.6.

(1) A non-zero linear functional $\alpha \in \mathfrak{h}'$ is called a root of the Lie algebra $\mathfrak{g}$ if there is a non-zero $X \in \mathfrak{g}$ such that

$$ad_H X = \alpha_H X \quad \text{for all} H \in \mathfrak{h}$$

(2) If α is a root, then the root space $\mathfrak{g}_\alpha$ is the eigenspace for root α

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \mathrm{ad}_H X = \alpha_H X\}$$

A nonzero element of $\mathfrak{g}_\alpha$ is called a root vector for α.

(3) The set

$$\Delta = \{\alpha \in \mathfrak{h}' | \alpha \text{ is a root}\}$$

is the collection of all the roots of $\mathfrak{g}$.

(4) The lattice generated by Δ(that, is all integer linear combinations of elements in Δ) is called the root lattice, Λ_R. ♣

Taking $\alpha = 0$, we see that $\mathfrak{g}_0$ is the set of all elements of $\mathfrak{g}$ that commute with every element of $\mathfrak{h}$. Since $\mathfrak{h}$ is a maximal commutative subalgebra, we conclude that $\mathfrak{g}_0 = \mathfrak{h}$. So the Cartan subalgebra is sometimes written as $\mathfrak{g}_0 = \mathfrak{h}$

Proposition 1.4.

Each root α, we have $\alpha \in i\mathfrak{t}$ ♠

Proof In particular, each $\mathrm{ad}_H, H \in \mathfrak{t}$, is skew self-adjoint(anti-Hermitian), which means that ad_H has pure imaginary eigenvalues. It follows that if α is a root, $\langle H, \alpha \rangle$ must be pure imaginary for $H \in \mathfrak{t}$, which can only happen if $\alpha \in i\mathfrak{t}$.

Proposition 1.5.

The Lie algebra can be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \tag{1.12}$$

and this is called the Cartan decomposition of $\mathfrak{g}$. ♠

Proposition 1.6.

1. If $\alpha \in \mathfrak{h}'$ is a root, so is $-\alpha$. Specifically, if $X \in \mathfrak{g}_\alpha$, then $X^* \in \mathfrak{g}_{-\alpha}$

2. The roots Δ span $\mathfrak{h}'$. ♠

Theorem 1.5.

For each root α, we can find linearly independent elements $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and H_α in $\mathfrak{h}$ such that H_α is a multiple of α and such that

$$[H_\alpha, X_\alpha] = 2X_\alpha$$

$$[H_\alpha, Y_\alpha] = 2Y_\alpha$$

$$[X_\alpha, Y_\alpha] = H_\alpha$$

Furthermore, Y_α can be chosen to equal X_α^* ♡

If X_α, Y_α and H_α are as in the theorem, then on the one hand, $[H_\alpha, X_\alpha] = 2X_\alpha$, while on the other hand, $[H_\alpha, X_\alpha] = \langle \alpha | H_\alpha \rangle X_\alpha$. Thus H_α is a multiple of α, we have

$$\langle H_\alpha | \alpha \rangle = 2 \Rightarrow H_\alpha = \frac{2\alpha}{\langle \alpha | \alpha \rangle} \tag{1.13}$$

And we calls H_α a coroot associated to the root α.

Proposition 1.7.

If X_α, Y_α and H_α are as in the theorem, and $Y_\alpha = X_\alpha^*$. Then we can construct

$$E_1^\alpha = \frac{i}{2} H_\alpha \qquad E_2^\alpha = \frac{i}{2}(X_\alpha + Y_\alpha) \qquad E_2^\alpha = \frac{1}{2}(X_\alpha - Y_\alpha)$$

are linearly independent elements of $\mathfrak{k}$ *and satisfy the commutation relations*

$$[E_i^\alpha, E_j^\alpha] = \epsilon_{ij}^k E_k^\alpha$$

So, the span of $E_1^\alpha, E_2^\alpha, E_3^\alpha$ *is a subalgebra of* $\mathfrak{k}$ *isomorphic to* $\mathfrak{su}(2)$

Proof The commutation relations is easy to check with the commutation relations of X_α, Y_α and H_α. All we need to do is to proof $E_1^\alpha, E_2^\alpha, E_3^\alpha \in \mathfrak{k}$.

Never mind, in physics we only need to put a factor i in front of Lie algebra X, then all the anti-Hermitian X will beccome Hermitian.

1.5 Structure of Cartan decomposition

Proposition 1.8.

Structure of the Cartan decomposition

1. $\Gamma|_{\mathfrak{h}\times\mathfrak{h}}$ *is non-degenerate.*
2. *If* $\alpha \in \Delta$*, and so is* $-\alpha$.
3. *For* $X \in \mathfrak{g}_\alpha$ *and* $Y \in \mathfrak{g}_{-\alpha}$*, we have* $[X, Y] = \Gamma(X, Y)H_\alpha$. *One can choose* X, Y *such that* $\Gamma(X, Y) = 1$.
4. $\dim(\mathfrak{g}_\alpha) = 1$ *for all* $\alpha \in \Delta$.
5. *Let* $\alpha \in \Delta$*, then from* $\{k\alpha | k \in \mathbb{Z}\}$*, only* α *and* $-\alpha$ *are roots.*
6. Δ *contains a basis of* $\mathfrak{h}'$
7. *For* $H_1, H_2 \in \mathfrak{h}$*, we have* $\Gamma(H_1, H_2) = \sum_{\alpha\in\Delta} \alpha_{H_1}\alpha_{H_2}$.

Based on this theorem, we can now construct the Cartan-Weyl basis of $\mathfrak{g}$. We start by choosing a basis $\{H_i\}$, where $i = 1, \cdots, r = \mathrm{rank}(\mathfrak{g})$, of the Cartan subalgebra(this is, of course, not unique). Also, we know that the eigenspaces $\mathfrak{g}_\alpha$ are one-dimensional for $\alpha \neq 0$, so we can choose $E_\alpha \in \mathfrak{g}$ such that $\mathfrak{g}_\alpha = \mathrm{span}(E_\alpha)$. In addition, these can be normalised so that $\Gamma(E_\alpha, E_{-\alpha}) = 1$. In summary, we have the basis

$$\{H_i, E_\alpha\} \quad i = 1, \cdots, r \text{ and } \alpha \in \Delta \tag{1.14}$$

We can describe α by a vector $(\alpha_1, \cdots, \alpha_r)$, where $\alpha_i = \alpha_{H_i}$. Now the Killing form, relative to the Cartan Weyl basis, then has components

$$\begin{aligned} \gamma_{ij} &= \sum_{\alpha\in\Delta} \alpha_i\alpha_j \quad \gamma_{i\alpha} = 0 \\ \gamma_{\alpha,-\alpha} &= 1 \qquad\qquad \gamma_{\alpha,\beta} = 0 \end{aligned} \tag{1.15}$$

Now we finally reach the the commutation relations for the Cartan-Weyl basis

$$\begin{aligned} [H_i, H_j] &= 0 \qquad\qquad [H_i, E_\alpha] = \alpha_i E_\alpha \\ [E_\alpha, E_{-\alpha}] &= H_\alpha = \alpha^i H_i \quad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta}E_{\alpha+\beta} & 0 \neq \alpha+\beta \in \Delta \\ 0 & 0 \neq \alpha+\beta \notin \Delta \end{cases} \end{aligned} \tag{1.16}$$

where $N_{\alpha\beta}$ are constants.

1.6 Weights

Weights

Definition 1.7.

For a representation $r : \mathfrak{g} \to \mathfrak{gl}(V)$, we call $w \in \mathfrak{h}'$ a weight of r if there is a non-zero vector $v \in V$ such that for all $H \in \mathfrak{h}$

$$r(H)v = w_H v$$

The eigenspace of a weight w, denoted by V_w, consists of all $v \in V$ which satisfy the eigen euqation.

♣

In other words, we are looking for common eigenvectors of the Cartan subalgebra, just as we have done for the adjoint representation. This means that the weights of the adjoint representation are the roots. The representation vector space V can be written as

$$V = \bigoplus_w V_w \tag{1.17}$$

where the sum runs over all weights of the representation r.

Ladder operators

How do the representation maps $r(E_\alpha)$ act on eigenvectors $v_w \in V_w$

$$\begin{aligned} r(H)r(E_\alpha)v_w &= (r(E_\alpha)r(H) + [r(H), r(E_\alpha)])v_w = (w_H r(E_\alpha) + r([H, E_\alpha]))v_w \\ &= (w_H r(E_\alpha) + r(\alpha_H E_\alpha))v_w = (w_H + \alpha_H)r(E_\alpha)v_w \end{aligned}$$

shows that $r(E_\alpha)v_w \in V_{w+\alpha}$. So, applying E_α to a vector with weight w leads to a vector with weight $w + \alpha$

Proposition 1.9.

If $r : \mathfrak{g} \to \mathfrak{gl}(V)$ is irreducible, then any two weights w_1, w_2 of r satisfy $w_1 - w_2 \in \Lambda_R$, that is, differences of weights are in the root lattice.

♠

Definition 1.8.

Let $r : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. A non-zero vector $v \in V$ is called a highest weight vector of r if $r(E_\alpha)v = 0$ for all $\alpha \in \Delta_+$. The weight λ of a highest weight vector is called highest weight.

♣