Chapter 1 Lie Groups and Lie Algebras

1.1 Manifolds

Given manifolds M_1, M_2, M_3 and maps $f : M_1 \to M_2$, $g : M_2 \to M_3$, the pullback of g under f is the map $f^*g : M_1 \to M_3$ defined by

$$f^*g = g \circ f \tag{1.1}$$

In particular, if $M_3 = \mathbb{R}$, the pullback of g under f is a function on M_1.

Given two manifolds M_1 and M_2 with a smooth map $f : M_1 \to M_2, p \mapsto q$, the pushforward of a vector $v \in T_pM_1$ is a vector $f_*v \in T_qM_2$ defined by

$$f_*v(g) = v(g \circ f) \tag{1.2}$$

for all smooth functions $g : M_2 \to \mathbb{R}$. Thus we can write

$$f_*v(g) = v(g \circ f) = v(f^*g) \tag{1.3}$$

Let ξ be a vector field on M. An integral curve of ξ is a differentiable curve $\gamma_\xi : [a, b] \to M$ such that

$$\partial_t\gamma_\xi(t) = T_t\gamma_\xi(\partial_t) = \xi_{\gamma(t)} \tag{1.4}$$

Provided an initial condition, such as $\gamma_\xi(0) = x$, is specified it has a unique solution. A flow combines all these solutions for different initial conditions.

The flow $\phi_t(x)$ of the vector field ξ on M is given by the unique integral curve $\gamma_\xi(t)$ with the initial condition $\phi_0(x) = x$. We have the following properties

1. $\phi_s \circ \phi_t = \phi_{s+t}$
2. $\partial_t\phi_t(x)|_{t=0} = \xi_x$

1.2 Lie Groups and Lie Algebras

Lie Groups

Definition 1.1.

A Lie group is a smooth manifold G which is also a group and such that the group product

$$G \times G \to G$$

and the inverse map is smooth. A connected Lie group is called an analytic group. ♣

A group representation is a group homomorphism $\rho : G \to GL(V)$. Since $GL(V)$ is also Lie group. Hence, we should think of representations of Lie groups G as Lie group morphisms $\rho : G \to GL(V)$ into the specific Lie group $GL(V)$, where Lie group morphisms means that

1. ρ is a differential map, maintains the differential manifold structure

2. ρ is a group homomorphisim, maintains the group structure

A vector field ξ on G is called left-invariant if TxLg(x) = gx for all x; g 2 G. Equivalently, this can also be written as g = .

Lie Algebras

Definition 1.2.

A Lie algebra is a vector space $\mathfrak{g}$ endowed with a bracket operation $[\cdot,\cdot]$ with the following properties

1. $[\cdot,\cdot]$ *is bilinear*
2. *Anti-symmetry:* $[X,Y] = -[Y,X]$
3. *Jacobi identity:* $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$

♣

Two elements X and Y of a Lie algebra $\mathfrak{g}$ commute if $[X,Y] = 0$. A Lie algebra $\mathfrak{g}$ is commutative if $\forall X, Y \in \mathfrak{g}$ we have $[X,Y] = 0$.

Representation of Lie Algebra

Let $\mathfrak{g}$ be a Lie algebra over k, and V a vector space over k, not necessarily finite-dimensional. A representation of a Lie algebra $\mathfrak{g}$ is a linear map $r : \mathfrak{g} \to \mathrm{End}(V)$,which maps $\mathfrak{g}$ into the vector space of all endomorphisms of V, $\mathrm{End}(V)$, such that

$$r([X,Y]) = [r(X), r(Y)]$$

for all $X, Y \in \mathfrak{g}$. The dimension of V is called the degree of r. r is said to be the trivial representation if dim $V = 1$ and $r(X) = 0$ for all $X \in \mathfrak{g}$.

Theorem 1.1. E

ery finite dimensional Lie algebra has a faithful matrix representation.

♡

Definition 1.3. A

inite-dimensional representation of a group or Lie algebra is said to be completely reducible if it is isomorphic to a direct sum of a finite number of irreducible representations.

♣

Structure Constants

In physics applications, it is common to introduce a basis $(\xi_1, \cdots, \xi_n)$ on a Lie algebra $\mathfrak{g}$. The commutator $[X_i, X_j] \in \mathfrak{g}$ must be a linear combination of these basis vectors.

$$[X_i, X_j] = {c_{ij}}^k X_k \tag{1.5}$$

The constants ${c_{ij}}^k$ are called the structure constants of the Lie algebra $\mathfrak{g}$. In such basis the representation can be written as

$$r([X_i, X_j]) = [r(X_i), r(X_j)] = {c_{ij}}^k r(X_k) \tag{1.6}$$

Lie Algebra of A Lie Group

A left translation, or right translation by an element $g \in G$ is the diffeomorphism

$$\begin{aligned} L_g &: G \to G \quad h \mapsto gh \\ R_g &: G \to G \quad h \mapsto hg^{-1} \end{aligned} \tag{1.7}$$

Left (or right) invariant if

$$L_g^* f = f \tag{1.8}$$

where L_g^* is the pull back of L_g. Of course, this just means that f is constant, because if $g = 1$ we have

$$L_g^* f(1) = L_e^* f(1) = f(e) \tag{1.9}$$

A vector field ξ on G is left (or right) invariant if

$$\begin{aligned} (L_g)_* \xi_x &= \xi_{gx} \\ (R_g)_* \xi_x &= \xi_{xg^{-1}} \end{aligned} \tag{1.10}$$

for all $g, x \in G$. And if we have two left invariant vector fields ξ, η on G, then

$$(L_g)_*[\xi, \eta] = [(L_g)_*\xi, (L_g)_*\eta] = [\xi, \eta] \tag{1.11}$$

Theorem 1.2. L

t G be a Lie group and $L(G)$ its Lie algebra. Then the map

$$\phi : L(G) \to T_1 G, \xi_x \mapsto \xi_1$$

is a linear isomorphism, or we can say $L(G) \cong T_1 G$. In particular

$$dim\ G = dim\ L(G)$$

♡

Proof Because the vector field ξ is left invariant,

$$(L_x)_* \xi_1 = \xi_x$$

the map ϕ

For all $g \in G$, we can use g^{-1} to move g to the indentity 1, and use $T_{g^{-1}}G$ to move the corresponding tangent sapce to the indentity 1. This means that the analytic structure is same everywhere on the manifold. Now we can define the Lie algebra $\mathfrak{g} = L(G)$ of a Lie group G to be the Lie algebra of left invariant vector fields on G.

1.3 The Exponential Map

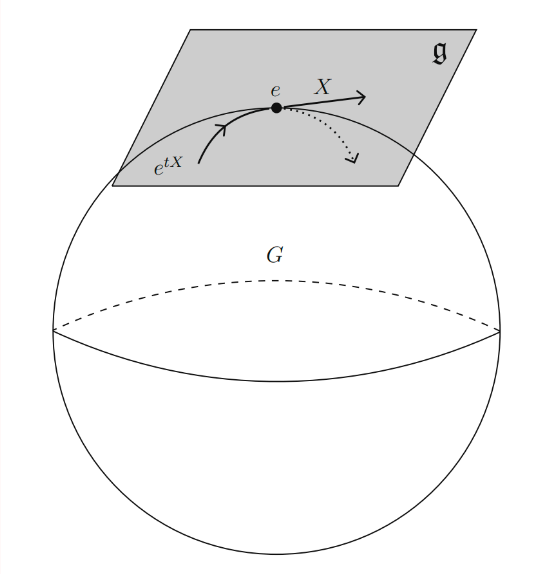

Figure 1.1: Lie algebra and exponential map

The Exponential Map

Motivations: consider an integral curve with the initial condition $\gamma(0) = g$, we can differentiate it at point g to get the corresponding tangent vector

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma(t) = X$$

Conversly we can define the inverse mapping, the exponential map, to get a point from a tangent vector, which means that we are going to define a map from the Lie algebra into the Lie group

$$\exp : L(G) \to G \tag{1.12}$$

and by Thm 1.2 we have $L(G) \cong T_1G$, so T_1G gives us an left invariant vector field ξ on G.

Definition 1.4.

The Lie algebra $L(G)$ gives us an left invariant vector field ξ on G with $v = \xi_1$. So we have an integral curve γ_ξ with initial condition $\gamma_v(0) = 1$. Then, the exponential map $\exp : L(G) \to G$(or $T_1G \to G$) is defined by

$$\exp(v) = \gamma_v(1) \tag{1.13}$$

♣

With the initial condition, we have

$$\gamma_v(s)\gamma_v(t) = \gamma_v(s+t) \qquad \gamma_{tv}(s) = \gamma_v(ts) \tag{1.14}$$

So for $\forall X \in L(G)$, we have the following properties

1. $\exp(tX)\exp(sX) = \exp[(s+t)X]$
2. $\exp(X)\exp(-X) = 1$
3. $\exp(0X) = 1$
4. $\partial_t|_{t=0} \exp(tX) = X$

The exponential map brings the structure fo addition in Lie algebra to group multiplication.

$$\partial_t|_{t=0} \exp(X+Y) = X+Y \tag{1.15}$$

Theorem 1.3. T

e exponential map has the following properties

1. *The exponential map is differentiable at the origin and* $T_0 \exp = \mathrm{id}_{L(G)}$
2. *The exponential map satisfies* $F \circ \exp = \exp' \circ T_1 F$ *for a group homomorphism* $F : G \to G'$

Proof (1)

One way to think about Lie algebras related to the classical groups is as infinitesimal group elements. Since Lie groups are manifolds we can consider a neighborhood of the identity element 1. In this neighborhood the manifold looks like a linear space, namely, the tangent space T_1G

1.4 The Adjoint Representation

Definition 1.5.

The adjoint representation of a Lie group G *is a representation* $Ad : G \to \mathfrak{g}$ *of the Lie group on its own Lie algebra* $\mathfrak{g} \cong T_eG$ *defined by*

$$Ad_g(x) = gxg^{-1}$$

The map $Ad : g \mapsto Ad_g$ *is the adjoint representation of* G

We can check the map is a homomorphisim

$$\mathrm{Ad}_{g_1g_2}(x) = g_1g_2x(g_1g_2)^{-1} = g_1(g_2xg_2^{-1})g_1^{-1} = \mathrm{Ad}_{g_1}\mathrm{Ad}_{g_2}(x) \tag{1.16}$$

Definition 1.6. T

e adjoint representation of Lie algebra $\mathfrak{g}$ *is a representation* $ad : \mathfrak{g} \to End(\mathfrak{g})$*, defined as*

$$ad = T_1 Ad$$

Theorem 1.4. I

$\mathfrak{g}$ *is a Lie algebra and* X *is an element of* $\mathfrak{g}$*, the representation of* $\mathfrak{g}$ *has the form*

$$ad_X(Y) = [X, Y]$$

for all $Y \in \mathfrak{g}$

Suppose we choose a basis $\{X_i\}$ on $L(G)$ and we want to work out the representation matrices of ad relative to this basis.

$$\mathrm{ad}_{X_i} X_j = [X_i, X_j] = {c_{ij}}^k X_k \tag{1.17}$$

hence, the representation matrices relative to this basis are given by the structure constants

$$[\mathrm{ad}_{X_i}]_j{}^k = {c_{ij}}^k \tag{1.18}$$

1.5 Lie's Theorem

Theorem 1.5.

The relation between Lie group and Lie algebra

1. *Every finite dimensional Lie algebra $\mathfrak{g}$ arises from a unique (up to isomorphism) connected and simply connected Lie group G*
2. *Under this correspondence, Lie group homomorphisms $\Phi : G_1 \to G_2$ are in 1 - 1 correspondence with Lie algebra homomorphisms $\phi : T_1G_1 \to T_1G_2$ such that*

$$\Phi(e^{tX}) = e^{t\phi(X)}$$

for all $t \in R$ and $X \in \mathfrak{g}$.

♡

Proof The proof was given by Lie, which is too complicated for me.

1.6 Matrix Lie Groups

Many of the Lie groups in physics are matrix Lie groups. If G is a matrix group, that is, a Lie subgroup of $GL(n, k)$ then the Lie algebra associated to the Lie group G is

$$L(G) = \{X = \partial_t|_0\gamma(t)|\gamma(t) \text{ integral curves with initial condition } \gamma(0) = 1\} \tag{1.19}$$

And we can use continuous parameter $t = (t^1, \cdots, t^k)$ to discribe the matrix Lie group.

For matrix we have a useful identity

$$\det(\exp(A)) = \exp(\text{Tr } A) \tag{1.20}$$

If G is a simply connected matrix Lie group with Lie algebra g, every representation of $L(G)$ comes from a representation of G.

Vector fields

Vector fields ξ on the matrix Lie group can be written as

$$\xi(t) = \xi^i(t)\partial_i = \xi^i(t)\frac{\partial g_\mu{}^\nu}{\partial t^i}\frac{\partial}{\partial g_\mu{}^\nu} = \xi^i(x)\text{Tr}\left(\frac{\partial g}{\partial t^i}\frac{\partial}{\partial g^T}\right) \tag{1.21}$$

where the last expression is just a short-hand for the one with indices in the middle

Generators

Motivation: Consider the definition of exponential maps

$$g(t) = \exp(tA) \tag{1.22}$$

So the generator of matrix Lie group G is defined by

$$T_i = \frac{\partial}{\partial t^i}\bigg|_{t=0} g(t) \tag{1.23}$$

and we can expand the group matrices near the identity as

$$g(t) = 1 + T_i t^i + O(t^2) \tag{1.24}$$

Lie Algebra of Matrix Lie Group

We just need to find the tangent space at identity

$$\xi(0) = \xi^i(x)\text{Tr}\left(\frac{\partial g}{\partial t^i}\frac{\partial}{\partial g^T}\right)\bigg|_{t=0} = \xi^i(0)\text{Tr}\left(T_i\frac{\partial}{\partial g^T}\right) \tag{1.25}$$

so we have

$$L(G) \cong T_1 G \cong \text{Span}\left(\{T_i\}\right) \tag{1.26}$$

the Lie algebra of matrix Lie group is the vector space of matrices spanned by the generators.

The Matrix Left-invariant Vector Fields

The left-translation, $L_g(x) = gx$ on x is just matrix multiplication

$$(gx)_\mu{}^\nu = g_\mu{}^\tau x_\tau{}^\nu \Rightarrow (T_x L_g)_{\mu\nu}{}^{\tau\sigma} = \frac{\partial (gx)_\mu{}^\tau}{\partial x_\sigma{}^\nu} = g_\mu{}^\rho \delta_\rho{}^\tau \delta_\nu{}^\sigma = g_\mu{}^\tau \delta_\nu{}^\sigma \tag{1.27}$$

For a left-invariant vector field ξ_g, we have $\xi_{gx} = T_x L_g \xi_x$. Consider

$$\xi^i(gx)\frac{\partial}{\partial t^i} = (T_x L_g)_{\mu\nu}{}^{\tau\sigma}\xi(x) \tag{1.28}$$

The Matrix Exponential

To translate the exponential map into the language of generators, we can consider a left-invariant vector field $\xi^i = v^j \xi^i_j$ with the associated generator $T = v^i T_i$. The integral curve $t_i = t_i(s)$ and $\gamma_v(s) = g(t(s))$ of this left-invariant vector field satisfy the differential equation

$$\frac{dt^i}{ds} = v_j \xi^j_i \Rightarrow \frac{d\gamma_v}{ds} = \frac{\partial g}{\partial t^i}\frac{dt^i}{ds} = T_i g v^i \tag{1.29}$$

The matrix exponential can be written as

$$e^X = \sum_n \frac{1}{n!}X^n$$

1.7 Complexifications

Definition 1.7.

If V is a finite-dimensional real vector space, then the complexification of V, denoted $V_{\mathbb{C}}$, is the space of formal linear combinations

$$v_1 + iv_2$$

with $V_1, v_2 \in V$ ♣

Mathematically, we could more pedantically define $V_{\mathbb{C}}$ to be the space of ordered pairs $(v_1, v2_2)$, but this is notationally cumbersome.

Proposition 1.1.

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then the bracket operation on $\mathfrak{g}$ has a unique extension to $\mathfrak{g}_{\mathbb{C}}$ that makes $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra. The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is called the complexification of the real Lie algebra $\mathfrak{g}$. ♠

Proof The uniqueness of the extension is obvious, since if the bracket operation on $\mathfrak{g}_{\mathbb{C}}$ is to be bilinear, then it must be given by

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1])$$

To show existence, we have to check the 3 properties of Lie algebra. But I do not want to write it down.

Proposition 1.2.

Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification, and $\mathfrak{h}$ an arbitrary complex Lie algebra. Then every real Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{h}$ extends uniquely to a complex Lie algebra homomorphism of $\mathfrak{g}_{\mathbb{C}}$ into $\mathfrak{h}$. ♠

Proof The unique extension is given by $\phi(X + iY) = \phi(X) + i\phi(Y)$. It is easy to check that this map is, indeed, a homomorphism of complex Lie algebras.