Groups and Subgroups

Groups: $G \times G \to G$, associativity, identity, inverse

Subgroups:

1. H is a subgroup $\Leftrightarrow \forall a, b \in H, ab^{-1} \in H$
2. Center: $C(G) = \{x \in G | \forall g \in G, gx = xg\}$
3. $|G| = 2 \Leftrightarrow G \cong \mathbb{Z}_2$

Cosets, Normal Subgroup & Quotient Group

Cosets: $aH = \{ah | h \in H\}$, $Ha = \{ha | h \in H\}$

1. Two left cosets are either disjoint or equal
2. G is divied by all the left(right) coset of H
3. $aH = bH$ iff $b \in aH$, $Ha = Hb$ iff $b \in Ha$
4. $|aH| = |bH| = |H|$ for all $a, b \in G$
5. Index $[G : H] = \#$ left(right) cosets
6. Lagrange Thm: $|G| = [G : H]|H|$
7. If $|G| = p$, G has no nontrivial subgroup

Normal subgroup: $aN = Na \Leftrightarrow aNa^{-1} = N$

Quotient Group: G/N, $(aN)(bN) = (ab)N$

Simple Group: no nontrivial normal subgroup

1. Subgroup of index 2 is normal
2. $N \lhd G$, $Q \le G$, $N \cap Q = \{e\}$, then $nq = qn$

Homomorphisms

Homomorphism: $\phi : G \to G', \phi(ab) = \phi(a)\phi(b)$

1. $\text{Ker}(\phi)$ is a noraml subgroup of G
2. $\text{Im}(\phi)$ is a subgroup of G'

Isomorphism: bijective homomorphism

1. ϕ an isomorphism $\Leftrightarrow \text{Ker}(\phi) = \{e\}, \text{Im}(\phi) = G'$
2. $G/\text{Ker}(\phi) \cong \text{Im}(\phi)$, $|G| = |\text{Ker}(\phi)||\text{Im}(\phi)|$

Group Actions

Group action: $\rho : G \times S \to S$

1. $ex = x$
2. $(g_1 g_2)x = g_1(g_2 x)$

Definitions

1. Orbit: $\mathcal{O}_x = \{gx | g \in G\}$
2. Stabilizer: $G_x = \{g \in G | gx = x\}$
3. g-fixed points: $Z_g = \{x \in S | gx = x\}$
4. Fixed points: $Z = \{x \in S | \forall g \in G, gx = x\}$
5. Quotient set: $S/G = \{\mathcal{O}_x | x \in S\}$

Properties

1. Two orbits are either disjoint or equal
2. $G_x = gG_y g^{-1} \Leftrightarrow \mathcal{O}_x = \mathcal{O}_y$
3. $(gG_x)x = gx$, $g_1 x \neq g_2 x \Leftrightarrow g_1 G_x \neq g_2 G_x$
4. G/G_x and $\mathcal{O}_x$ is 1-1 correspondence
5. $|S| = \sum_i |\mathcal{O}_{x_i}| = \sum_i [G : G_{x_i}]$
6. $\#$ orbits $= \frac{1}{|G|} \sum_{g \in G} |Z_g|$

Adjoint/Conjugation

Group action on itself:

1. Left multiplication: $\text{L}_g(x) = gx$
2. Adjoint/Conjugation: $\text{Ad}_g(x) = gxg^{-1}$
3. Calyey: Any finite group G is isomorphic to a subgroup of S_n with $n = |G|$

Definitions

1. Conjugacy calss: $[x]_H = \{hxh^{-1} | h \in H\}$
2. Centralizer: $C_H(x) = \{h \in H | hxh^{-1} = x\}$
3. Center: $C(G) = \{x \in G | \forall g \in G, gxg^{-1} = x\}$

Properties

1. $|S| = \sum_i |[x]_G| = \sum_i [G : C_G(x_i)]$
2. $a, b \in [x]$, then $|a| = |b|$
3. $N \lhd G$, then N is a union of conjugacy calsses
4. Ad_g and $\text{L}_g(x)$ induce Aut of G.
5. Hom $\text{Ad} : G \to \text{Aut}(G)$, $\ker(\text{Ad}) = C(G)$
6. $\#$ conjugacy calsses $= \frac{1}{|G|} \sum_{g \in G} |C_G(g)|$
7. $|[g]| = 1 \Leftrightarrow g \in C(G)$

Automorphisms: isomorphism onto itself

1. Inner Aut: $\text{Inn}(G) = \{\text{Ad}_g | g \in G\}$
2. Outer Aut: $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$
3. G is abelian $\Leftrightarrow \text{Inn}(G) = \{\text{id}\}$
4. Hom $\text{Ad} : G \to \text{Aut}(G)$, $\text{Ker}(\text{Ad}) = C(G)$

Schur's Lemma

If V and W are Irreps of G and $\phi : V \to W$ is a linear map such that $\rho_V \phi = \phi \rho_W$, then

1. Either ϕ is an isomorphism. or $\phi = 0$
2. If $V = W$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$

Corollary: All Irreps of abelian group are 1-dim.

Direct Products & Semidirect Products

Direct product: $G \times H = \{(g, h) | g \in G, h \in H\}$

Semidirect product: $N \lhd G$, $Q \le G$, if Hom $G \to G/N$ induces an Iso $Q \to G/N$, then $G = N \rtimes_\phi Q$ with $\phi : Q \to \text{Aut}(N)$ a Hom

1. If m, n is comprime, then $C_{mn} = C_m \times C_n$
2. $G = N \rtimes_\phi Q \Leftrightarrow N \lhd G$, $NQ = G$, $N \cap Q = \{e\}$
3. $N, Q \lhd G$, semidirect product $\to$ direct product
4. $(n, h)(n', h') = (n\phi_h(n'), hh')$
5. $(n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$

Representations

Reps

1. Rep of G in V is a Hom $\rho : G \to GL(V)$
2. Subrep: an invariant subspace W of V
3. $\dim(V \oplus W) = \dim(V) + \dim(W)$
4. $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$

Irreps

1. Irrep $\Leftrightarrow V$ has no nontrivial invariant subspace
2. Every Rep is a direct sum of Irreps
3. $V = \alpha_1 V_1 \oplus \cdots \oplus \alpha_k V_k$
4. Equivilant $\Leftrightarrow \mathcal{R}' = A\mathcal{R}A^{-1}$

Characters

Group function: $f : G \to \mathbb{C}$

Class function: $f(gh) = f(hg)$

Character: $\chi(g) = \text{Tr}(\rho_g)$

1. $\chi_\rho(e) = \dim V = n$
2. $\chi_\rho(g^{-1}) = \chi_\rho^*(g)$
3. $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$
4. $\chi_{V \oplus W} = \chi_V + \chi_W$
5. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
6. $\rho_V \cong \rho_V \Leftrightarrow \chi_V = \chi_V$

Compleness of Characters

Characters of Irreps of G $\{\chi_i\}_{i=1}^k$ form an orthonormal basis for $Cl(G)$ (vector space of clsss functions)

1. $\#$ Irreps of $G = \#$ conjugacy classes of G
2. $\frac{1}{|G|} \sum_i \chi_i^*(g)\chi_i(g) = \frac{1}{|[g]|}$
3. $\frac{1}{|G|} \sum_i \chi_i(g)^* \chi_i(h) = 0$ for $[g] \neq [h]$

Orthogonality Relations for Characters

Inner product: $\langle\phi|\psi\rangle = \frac{1}{|G|}\sum_{g\in G}\phi^*(g)\psi(g)$

If χ and χ' are the characters of two nonisomorphic Irreps, we have

1. $||\chi||^2 = \langle\chi|\chi\rangle = 1$
2. $\langle\chi|\chi'\rangle = 0$

If $V = \alpha_1 V_1 \oplus \cdots \oplus \alpha_k V_k$, we have

1. Rep is determined by its character
2. The multiplicity α_i of V_i in V is $\alpha_i = \langle\chi_i|\chi\rangle$
3. A Rep V is irreducible iff $\langle\chi|\chi\rangle = 1$

Matrix Elements of Representation

Matrix element: $(\rho_V)_{ij} : G \to \mathbb{C}$

1. Orthogonality: $\langle\rho_{ij}|\rho'_{i'j'}\rangle = \frac{1}{\dim(\rho)}\delta_{\rho\rho'}\delta_{ii'}\delta_{jj'}$
2. Compleness: $\frac{1}{|G|}\sum_{\rho,ij}\dim(\rho)\rho^*(g)\rho(h) = \delta_{gh}$

Matrix element of all Irreps $\{\rho_{ij}^{(\alpha)}\}$ form an orthonormal basis for $V(G)$ (vector space of group functions)

Regular Representations

Group algebra A_G: The set of formal linear combinations $v = \sum_{g\in G} c_g g$

1. $\dim A_G = |G|$
2. Regular Rep is $R(g)v = gv = \sum_{s\in G} c(s)(gs)$
3. Regular Rep is in forms of permutation matrix
4. Regular Rep is faithful (injective)

Decomposition of Regular Representations

1. Character r_G: $r_G(e) = |G|, r_G(g) = 0\ (g \neq e)$
2. Every Irrep V_i is contained in regular Rep R
3. Multiplicity of V_i is equal to its degree n_i
4. $\sum_i n_i^2 = |G|$
5. $\sum_i n_i\chi_i(g) = 0$ for $g \neq e$

Decomposition of Tensor Products

Tensor product of 2 Irreps V_i, V_j

1. $V_i \otimes V_j = \oplus_\alpha {N_{ij}}^\alpha V_\alpha$
2. ${N_{ij}}^\alpha = \langle\chi_\alpha|\chi_{i\otimes j}\rangle = \frac{1}{|G|}\sum_{g\in G}\chi^*_\alpha(g)\chi_i(g)\chi_j(g)$
3. ${N_{ij}}^\alpha = {N_{ji}}^\alpha$
4. Matrix ${(T_i)_j}^k = {N_{ij}}^k$, then $[T_i, T_j] = 0$

Symmetric Groups S_n

Basic properties

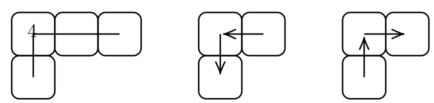
1. $|S_n| = n!$, $S_n = A_n \rtimes C_2$
2. $A_n = \{\text{All even permutations}\}$
3. $(12)(23) = (123)$, $\tau(123)\tau^{-1} = (\tau(1)\tau(2)\tau(3))$
4. S_n can be generated by $(12)(23)\cdots$(n-1,n)
5. Center: $C(S_n) = \{e\}$ for $n \geq 3$
6. Nontrivial normal subgroup: A_n, $n = 3$, $n \geq 5$

Conjugacy classes

1. Conjugacy class $[\mu] \Leftrightarrow$ Integer partitions μ
2. $\mu = (\mu_1, \cdots, \mu_k)$, $n = \sum_{i=1}^k \mu_i$, $\mu_1 \geq \cdots \geq \mu_k$
3. $|[\mu]| = \frac{n!}{\prod_r \mu_r^{m_r} m_r!}$, m_r: multiplicity of μ_r

Reps

1. Irrep $V_\mu \Leftrightarrow$ Conjugacy class $[\mu]$
2. dim $V_\mu = \#$ Standard Young tableaux Y_μ
3. $\#$ Standard Young tableaux $Y_\mu = \frac{n!}{\prod_{s\in Y(\mu)} h_\mu(s)}$
4. Form of Irrep $\Leftrightarrow$ Young operator $E(q)$
5. Rep of (k,k+1)
   - (a) Basis: Standard Young tableaux $Y_r^{(\mu)}$
   - (b) $k,\ k+1$ in the same row: $(k,k+1)_{rr} = 1$
   - (c) $k,\ k+1$ in the same col: $(k,k+1)_{rr} = -1$
   - (d) $(k,k+1)Y_r^{(\mu)} = Y_s^{(\mu)}$ with ρ:
     $(k,k+1)_{rr} = -\rho$, $(k,k+1)_{ss} = \rho$
     $(k,k+1)_{rs} = (k,k+1)_r = \sqrt{1-\rho^2}$



$h_\mu(1) = 4$ $\quad$ $1/\rho = 2$ $\quad$ $1/\rho = -2$

Cyclic Group C_n

$C_n = \langle a\rangle = \{e, a, \cdots, a^{n-1}\}$, Abelian

Irreps:

1. $\rho(a^m) = w^m = (e^{\frac{2\pi ik}{n}})^m$, $k = 0, 1, \cdots, n-1$
2. $\#$ Irreps $= \#$ Conjugacy classes $= |C_n|$

Character Table of C_3

	e	a	a^2
χ_1	1	1	1
χ_2	1	w	w^2
χ_3	1	w^2	w

Symmetric Groups S_3

1. $S_3 = \{e, (12), (23), (13), (123), (132)\}$
2. $x = (123)$, $y = (12)$, $x^3 = y^2 = (yx)^2 = e$
3. $yxy = x^{-1} = x^2$
4. $S_3 \cong D_3$

Irreps of S_3

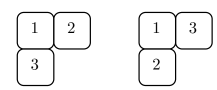
1. Trivial Rep: $\rho_1(g) = 1$
2. $\rho_2(e) = \rho_2(123) = \rho_2(132) = 1$
   $\rho_2(12) = \rho_2(13) = \rho_2(23) = -1$
3. 2-dim Irrep:

$$\rho_3(e) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \rho_3(123) = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix} \quad \rho_3(132) = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$$

$$\rho_3(12) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad \rho_3(23) = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \quad \rho_3(13) = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

Character Table of S_3

	$[(e)]$	$[(12)]$	$[(123)]$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	1



$(23)_{11} = -1/2$ $\quad$ $(12)_{11} = 1$

$(23)_{22} = 1/2$ $\quad$ $(12)_{22} = -1$

$(23)_{12} = (23)_{21} = \sqrt{3}/2$ $\quad$ $(12)_{12} = (12)_{21} = 0$

Lie Groups & Lie Algebras

Lie group G: manifold + group structure

Lie algebra $\mathfrak{g}$: vector space + Lie braket

Lie Thm

1. Lie group $G \Leftrightarrow$ Lie algebra $L(G)$
2. $\exp : L(G) \to G$, $\frac{d}{dt}|_{t=0} : G \to L(G)$
3. Lie group Hom $\Phi \Leftrightarrow$ Lie algebra Hom ϕ
4. $\frac{d}{dt}|_{t=0}\exp(itX) = X$, $\Phi(e^{tX}) = e^{t\phi(X)}$

Compact Lie Group:

1. For physics, compact = bounded and closed
2. U(1) = $\mathbb{R}$, not compact
3. SO(2) = S^1, SU(2) = S^3, compact
4. U(1)/$\mathbb{Z} \Rightarrow$ Compact!

Examples:

1. $GL(n) = \{A \in M_n |\det A \neq 0\}$
2. $SL(n) = \{A \in M_n |\det A = 1\}$
3. $\mathfrak{gl}(n) = M_n$, $\mathfrak{sl}(n) = \{A \in \mathfrak{gl}(n)|\text{Tr } A = 0\}$

Dihedral Groups D_n

D_n: R ratation, r reflection

1. $D_n = \langle R, r\rangle = \langle R\rangle \rtimes \langle r\rangle = C_n \rtimes C_2$
2. $R^n = r^2 = (rR)^2 = e$
3. $R^n r R^n = r$, $rR^n r = R^{-n}$

Conjugacy classes for $n = 2m$

1. $[e] = \{e\}$
2. $\{R^k, R^{-k}\}\&\{R^m\}$, $(k = 1, \cdots, m-1)$
3. $\{rR^{2k} | k = 1, \cdots, m\}$
4. $\{rR^{2k+1} | k = 1, \cdots, m-1\}$
5. # Conjugacy classes $= m + 3$

Conjugacy classes for $n = 2m+1$

1. $[e] = \{e\}$
2. $\{R^k, R^{-k}\}$, $(k = 1, \cdots, m)$
3. $\{rR^{2k} | k = 1, \cdots, 2m\}$
4. # Conjugacy classes $= m + 2$

Center of D_n $(|[g]| = 1)$

1. $n = 2m$, $C(D_n) = \{e, R^m\}$
2. $n = 2m+1$, $C(D_n) = \{e\}$

Non-trivial normal subgroup for $n = 2m$

1. $\langle R^k\rangle$ with k divie n
2. $\langle R^2, r\rangle$ with order m
3. $\langle R^2, rR\rangle$ with order m

Non-trivial normal subgroup for $n = 2m+1$

1. $\langle R^k\rangle$ with k divie n

Dihedral Groups D_2

1. $D_2 = \{e, R, r, rR\}$, Abelian
2. $Rr = rR$, Abelian
3. Conjugacy classes: $[e]$, $[R]$, $[r]$, $[rR]$
4. $D_2 \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Irreps: ± 1 correspond to R and r in all possible ways

Character Table of D_2

	e	R	r	rR
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Simple

Simple Lie algebra:

1. Has no non-trivial ideal
2. Non-abelian

Simple Lie group:

1. Has no non-trivial normal subgroup
2. Connected, Non-abelian

Examples: $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{u}(3)$, SL(2)

Structure Constants

Structure constant ${f_{ab}}^c$: $[X_a, X_b] = i{f_{ab}}^c X_c$

1. Structure constants is a tensor
2. ${f_{ij}}^l {f_{kl}}^n + {f_{jk}}^l {f_{il}}^n + {f_{ki}}^l {f_{jl}}^n = 0$
3. Matrix ${(T_i)_j}^k = {f_{ij}}^k$, then $[T_i, T_j] = i{f_{ij}}^k T_k$
4. $A = A^i X_i$, $[A, B]^i = i{f_{jk}}^i A^j B^k$
5. Under linear transformation $X_i \to X'_i = {T_i}^j X_j$
   $\Rightarrow {f_{ij}}^k \to {f'_{ij}}^k = {T_i}^l {T_j}^m {(T^{-1})_n}^k {f_{lm}}^n$

Adjoint Representations

Adjoint Rep of G: Ad:$G \to L(G)$, $g \mapsto \mathrm{Ad}_g$

Adjoint Rep of $\mathfrak{g}$: ad:$\mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$, $X \mapsto \mathrm{ad}_X$

1. Ad_g is an operator of A_G, $\mathrm{Ad}_g(x) = gxg^{-1}$
2. ad_g is an operator of $\mathfrak{g}$, $\mathrm{ad}_X(Y) = [X, Y]$
3. $\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$

Structures of ad $(A = A^i X_i)$

1. ${(\mathrm{ad}_A)_j}^i B^j = [A, B]^i = i{f_{ik}}^i A^j B^k$
   $\Rightarrow {(\mathrm{ad}_A)_j}^i = i{f_{jk}}^i A^k$
2. Killing form: $\gamma_{ij} = {f_{ik}}^l {f_{jl}}^k$
3. ${f_{ij}}^k = -{f_{ji}}^k$, $f_{ijk} = \gamma_{kl} {f_{ij}}^l$ is totally ant-sym

O(n) & SO(n)

1. $O(n) = \{A \in GL(n) | AA^T = I\}$
2. $SO(n) = \{A \in O(n) | \det A = 1\}$
3. $\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in \mathfrak{gl}(n) | A^T = -A\}$
4. $\dim O(n) = \dim SO(n) = \frac{1}{2}n(n-1)$
5. $SO(n) \triangleleft O(n)$, $O(n) = SO(n) \rtimes \mathbb{Z}_2$
6. $C(O(n)) = \{I, -I\} \cong \mathbb{Z}_2$

U(n) & SU(n)

1. $U(n) = \{A \in GL(n) | AA^\dagger = I\}$
2. $SU(n) = \{A \in U(n) | \det A = 1\}$
3. $\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n) | A^\dagger = -A\}$
4. $\mathfrak{su}(n) = \{A \in \mathfrak{gl}(n) | A^\dagger = -A, \mathrm{Tr}(A) = 0\}$
5. $C(SU(n)) \cong \mathbb{Z}_n$ with the form $e^{i\theta} I$
6. $\mathbb{Z}_k \triangleleft U(1) \triangleleft SU(n) \triangleleft U(n)$
7. $\dim U(n) = n^2$, $\dim SU(n) = n^2 - 1$

Baker Campbell Hausdorff Formular

1. $\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$
2. $e^X Y e^{-X} = \sum \frac{1}{n!}[X^{(n)}, Y]$, $X^{(n+1)} = [X^{(n)}, Y]$
3. $e^{-tA(x)} \frac{d}{dx} e^{tA(x)} = \int_0^t e^{-sA} \frac{dA}{dx} e^{sA} ds$
4. $\frac{d}{dx} e^{A(x)} = \int_0^1 e^{(1-s)A} \frac{dA}{dx} e^{sA} ds$
5. $e^X e^Y = e^Z$
6. $Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [[X, Y], Y]) + \cdots$

Roots and Weights

Cartan Subalgebra $\mathfrak{h}$, for all $H \in \mathfrak{h}$

1. ad_H can be simultaneously diagonalised
2. $\mathfrak{h}$ is maximal, $\mathfrak{h}$ is Abelian

Roots

1. $\mathrm{ad}_H X = \langle \alpha | H\rangle X$ for all $H \in \mathfrak{h}$, α is root
2. $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \mathrm{ad}_H X = \langle \alpha | H\rangle X\}$
3. $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_\alpha \mathfrak{g}_\alpha)$
4. If α is a root, then so is $-\alpha$
5. If $X \in \mathfrak{g}_\alpha$, then $X^* \in \mathfrak{g}_{-\alpha}$

Weights: For a representation $\mathcal{R} : \mathfrak{g} \to \mathfrak{gl}(V)$

1. $\mathcal{R}(H)v = \langle w | H\rangle v$ for all $H \in \mathfrak{h}$, w is weight
2. Weights of adjoint Rep are the roots
3. $V = \oplus_w V_w$
4. $\mathcal{R}_H \mathcal{R}_{E_\alpha} |w\rangle = (\langle \alpha | H\rangle + \langle w | H\rangle) \mathcal{R}_{E_\alpha} |w\rangle$
5. $\mathcal{R}_H \mathcal{R}_{E_\alpha} |w\rangle = (\alpha_H + w_H) |w + \alpha\rangle$

Sub SO(3)

$$S(\mathbf{n}) = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Cartan-Weyl Basis

Commutation relations for the Cartan-Weyl basis

1. $[H_i, H_j] = 0$, $H_i^* = H_i$
2. $[H_i, E_\alpha] = \alpha_i E_\alpha$, $E_\alpha^* = E_{-\alpha}$
3. $[E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^i H_i$

Normalized Cartan-Weyl basis

1. $\mathrm{Tr}(E_\alpha E_\beta) = \lambda \delta_{\alpha\beta}$
2. For $\mathfrak{su}(2)$, $E_\pm = \frac{1}{\sqrt{2}} J_\pm$

$\mathfrak{su}(2)$ Subalgebra

1. $\pm\alpha$ of semi-simple Lie algebra $\Leftrightarrow$ $\mathfrak{su}(2)$
2. $E_\pm = \frac{1}{|\alpha|} E_{\pm\alpha}$, $H = \frac{1}{|\alpha|^2} H_\alpha$

SU(2)

Geo Rep of SU(2)

1. SU(2)= S^3, 4-dim spherical coordinates (ω, θ, ϕ)
2. $x_1 = \sin\frac{\omega}{2}\sin\theta\cos\phi$, $x_2 = \sin\frac{\omega}{2}\sin\theta\sin\phi$
   $x_3 = \sin\frac{\omega}{2}\cos\theta$, $x_4 = \cos\frac{\omega}{2}$
3. $\omega \in [0, 2\pi]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$
4. North pole: $\omega = 0$, South pole: $\omega = 2\pi$
5. $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \Rightarrow \omega\mathbf{n}$
6. SU(2) $\Leftrightarrow S^3 \Leftrightarrow B^3_{2\pi}$ 3-dim ball $r = 2\pi$

Exp Reps of SU(2)

1. $U(\mathbf{n}, w) = \exp(\frac{i}{2} w \mathbf{n}\cdot\boldsymbol{\sigma}) = \cos\frac{w}{2} I + i\sin\frac{w}{2}\cdot(\mathbf{n}\cdot\boldsymbol{\sigma})$
2. $U(\mathbf{n}, w + 2\pi) = -U(\mathbf{n}, w)$, $U(-\mathbf{n}, w) = U(\mathbf{n}, -w)$

Conjugacy calsses of SU(2)

1. For an ω, $\{U(\mathbf{n}, \omega) | \forall \mathbf{n}\}$ is a conjugacy calss
2. Class function on SU(2) only depend on ω

SO(3)

Geo Rep of SO(3)

1. Upper-half SU(2) sphere, $\omega \in [0, \pi]$
2. SO(3) $\Leftrightarrow$ Half of $S^3 \Leftrightarrow B^3_\pi$ 3-dim ball $r = \pi$
3. $R(\mathbf{n}, \pi) = R(-\mathbf{n}, \pi)$ Antipodal points $\Rightarrow$ Same

Exp Reps of SO(3)

1. $R(\mathbf{n}, w) = \exp(iw\mathbf{n}\cdot\mathbf{T})$
2. $R(\mathbf{n}, w + 2\pi) = R(\mathbf{n}, w)$, $R(-\mathbf{n}, w) = R(\mathbf{n}, -w)$

Properties

1. $S(\mathbf{n}) T_3 S^{-1}(\mathbf{n}) = \sum_i n_i T_i$, $S(\mathbf{n})\hat{e}_3 = \mathbf{n}$
2. $SR(\mathbf{n}, w)S^{-1} = (S\mathbf{n}, w)$
3. For an ω, $\{R(\mathbf{n}, \omega) | \forall \mathbf{n}\}$ is a conjugacy calss

SU(2) & SO(3)

Relations

1. For $X \in \mathfrak{su}(2)$, $U \in$SU(2), write $X = \mathbf{x}\cdot\boldsymbol{\sigma}$
2. $\mathrm{ad}_U X = UXU^{-1} = X' = \mathbf{x}'\cdot\boldsymbol{\sigma}$
3. Define $\mathbf{x}' = D(U)\mathbf{x}$, then D : SU(2)$\rightarrow$SO(3)
4. Double covering: $D(U(\mathbf{n}, w)) = R(\mathbf{n}, w)$
5. SU(2)/ $\mathbb{Z}_2$ = SO(3)

Subgroups of SU(2) and SO(3)

1. $\mathrm{U}(1)_\mathbf{n} = \{U(\mathbf{n}, w) | w \in [0, 2\pi)\} \leq \mathrm{SU}(2)$
2. $\mathrm{SO}(2)_\mathbf{n} = \{R(\mathbf{n}, w) | w \in [0, 2\pi)\} \leq \mathrm{SO}(3)$
3. $\mathbb{Z}_k \lhd U(1) \lhd SU(n) \lhd U(n)$
4. $D(\mathbb{Z}_k) = \mathbb{Z}_k$(odd) or $\mathbb{Z}_{k/2}$(even)
5. SU(2)/U(1) = S^2, Inn(SU(2)) = SO(3)

$\mathfrak{su}(2)$

Properties

1. $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, $[J_i, J_j] = i\epsilon_{ij}{}^k J_k$
2. Irreps: V_j, $2j \in \mathbb{Z}^+$, $\dim V_j = 2j + 1$
3. Cartan generator: $H = J_3$, rank = 1
4. Roots: $\Delta_\alpha = \{\pm 1\}$
5. Cartan-Weyl basis: $E_\pm = J_1 \pm iJ_2$, $H = J_3$

2-dim generators Rep: $J_i = \sigma_i/2$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3-dim generators Rep: $J_i = T_i/2$

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, T_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathfrak{su}(3)$

1. $[T_i, T_j] = i\epsilon_{ij}{}^k T_k$
2. Cartan generator: $H_1 = \lambda_3$, $H_2 = \lambda_8$, rank = 2
3. Roots: $\Delta_\alpha = \{\pm(1, 0), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$
4. Cartan-Weyl basis: $E_{\pm\alpha_1} = T_1 \pm iT_2$, $E_{\pm\alpha_2} = T_4 \pm iT_5$, $E_{\pm\alpha_3} = T_6 \pm iT_7$, $H_1 = T_3$, $H_2 = T_8$

3-dim Rep: $T_i = \lambda_i/2$ (Gell-Mann matrices λ_i)

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Finite Subgroups of SO(3)

SO(3) Acts on $\mathbb{R}^3$

1. Orbit of $\mathbf{n}$: $\mathcal{O}_\mathbf{n} = \{\mathbf{x} : |\mathbf{x}| = |\mathbf{n}|\}$, a sphere
2. Stabilizer of $\mathbf{n}$: $G_\mathbf{n} = \{R(\mathbf{n}, w) | w \in [0, 2\pi)\}$

Poles

1. $P_g = Z_g = \{N_g, S_g\}$, $|P_g^G| = 2$
2. $P_G = \{P_g | \forall g \in G\}$
3. $\forall g \in G$, $\forall x \in P_G$, $gx \in P_G$
4. $\forall x \in P_G$, $2 \leq |G_x| \leq |G|$

Finite subgroups of SO(3)

1. $|P_G| = 2$: Rotation with a fixed axi, C_n
2. $|P_G| = 3$: $\frac{1}{|G_{x_1}|} + \frac{1}{|G_{x_2}|} + \frac{1}{|G_{x_3}|} = 1 + \frac{2}{|G|}$

Class	$(\lvert G_{x_1}\rvert, \lvert G_{x_2}\rvert, \lvert G_{x_3}\rvert)$	G	Polyhedra
D_{n+2}	$(2, 2, n)$	D_n	Dihedra
E_6	$(2, 3, 3)$	A_4	4
E_7	$(2, 3, 4)$	S_4	6, 8
E_8	$(2, 3, 5)$	A_5	12, 20

Euler number: $\chi = V - E + F = 2$

Groups and Q.M.

$\mathcal{H}$ is Hilbert space and H is Hamiltonian

1. G is group of symmetry
2. A unitary or antiunitary Rep $U(G, \mathcal{H})$
3. $[U(g), H] = 0$ for all $g \in G$
4. $HU(g)|\psi\rangle = U(g)H|\psi\rangle = EU(g)|\psi\rangle$
5. $U(g)|\psi\rangle$ and $|\psi\rangle$ may be the same state

If $U(G, \mathcal{H})$ has n Irreps V_i, s.t. $\mathcal{H} = \oplus_\alpha m_\alpha V_\alpha$

1. $\Psi_i = \{U_i(g)|\psi\rangle : \forall g \in G\}$, $V_i = \mathrm{span}(\Psi_i)$
2. V_i is an eigenspace of H
3. H has at most $\sum_\alpha m_\alpha$ eigenvalues in $\mathcal{H}$

Proof: Suppose $|\phi\rangle \in V_i$ and $|\phi\rangle \notin \mathrm{span}(\Psi_i)$, we must have $V = \mathrm{span}(\Psi_i)$, $V_i = V \oplus V^\perp$ where $|\phi\rangle \in V^\perp$. But $V^\perp$ and V are both invariant under $U(g)$, which means that $U(G, \mathcal{H})$ is reducible. This leads to contradiction!

Pauli Matrix

1. $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ij}{}^k\sigma_k$
2. $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k\sigma_k$, $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$
3. $(\mathbf{X}\cdot\boldsymbol{\sigma})(\mathbf{X}\cdot\boldsymbol{\sigma}) = |X|^2 I$
4. $\sum_a (\sigma_a)_{ij}(\sigma_a)_{kl} = -\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk}$